The Aharonov-Anandan phase of a classical dynamical system seen mathematically as a quantum dynamical system

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It is shown that the non-adiabatic Hannay's angle of an integrable non-degenerate classical hamiltonian dynamical system may be related to the Aharonov-Anandan phase it develops when it is looked mathematically as a quantum dynamical system.

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Contents

I. A QUANTUM DYNAMICAL SYSTEM SEEN MATHEMATICALLY AS A CLASSICAL DYNAMICAL SYSTEM

Let us start from the following:

DEFINITION I.1

quantum dynamical system:
a couple $(\mathcal{H}, \hat{H})$ such that:

- $\mathcal{H}$ is an Hilbert space
- $\hat{H} \in \mathcal{L}_{s.a.}(\mathcal{H})$ (called the quantum hamiltonian)

where $\mathcal{L}_{s.a.}(\mathcal{H})$ denotes the set of all the self-adjoint linear operators over $\mathcal{H}$.

Given a quantum dynamical system QDS = $(\mathcal{H}, \hat{H})$ let us introduce the following:

DEFINITION I.2

classical hamiltonian dynamical system associated to QDS

$$CDS[QDS] \; := \; ((\mathcal{P}(\mathcal{H}), \omega_{Kahler}[g_{Fubini-Study}]), H)$$

where:

- $P(\mathcal{H}) \; := \; \frac{\mathcal{H}}{\sim}$ is the projective Hilbert space associated to $\mathcal{H}$ i.e. the set of equivalence classes in $\mathcal{H}$ with respect to the following equivalence relation:
$$|\psi> \sim |\phi> := \; \exists c \in \mathbb{C} \; : \; |\psi> = c|\phi>$$
- $\omega_{Kahler}[g_{Fubini-Study}]$ is the Kähler form of the Fubini-Study metric over $\mathcal{P}(\mathcal{H})$ [1]
- $H \in C^{\infty}(\mathcal{P}(\mathcal{H}))$ is the hamiltonian defined by:
$$H(P_{\psi}) \; := \; Tr(\hat{H} P_{\psi}) \tag{1.1}$$

$P_{\psi} := |\psi> < \psi|$ being the projector associated to a $|\psi> \in \mathcal{S}(\mathcal{H})$, where $\mathcal{S}(\mathcal{H}) \; := \; \{|\psi> \in \mathcal{H} \; : < \psi|\psi> = 1\}$ is the unit sphere in $\mathcal{H}$ and where we have used the fact that:

$$\mathcal{P}(\mathcal{H}) \; \sim_{diff} \; \{P_{\psi}, |\psi> \in \mathcal{S}(\mathcal{H})\} \tag{1.2}$$

Remark I.1

Since $(\mathcal{P}(\mathcal{H}), g_{Fubini-Study})$ is a Kähler manifold, its Kähler form $\omega_{Kahler}[g_{Fubini-Study}]$ is in particular a symplectic form, so that CDS [QDS] is indeed a classical hamiltonian dynamical system [2].

Remark I.2

The classical hamiltonian dynamical system CDS [QDS] has not to be confused with the classical dynamical system obtained taking the classical limit of QSD since obviously:

$$CDS[QDS] \; \neq \lim_{\hbar \rightarrow 0} QDS \;\; \forall QDS \tag{1.3}$$

CDS[QDS] is the quantum dynamical system QDS seen mathematically as a classical hamiltonian dynamical system.

Remark I.3

The projective unitary group $U(\mathcal{P}(\mathcal{H})) \; := \; \frac{U(\mathcal{H})}{U(1)}$ of $\mathcal{H}$ acts on $\mathcal{P}(\mathcal{H})$ by isometries of $g_{Fubini-Study}$ that are symplectomorphisms of the symplectic manifold $(\mathcal{P}(\mathcal{H}), \omega_{Kahler}[g_{Fubini-Study}])$.

Let $\rho : G \mapsto \mathcal{P}(\mathcal{H})$ be a projective unitary representation on $\mathcal{H}$ of a Lie group G. The associated momentum map $\mathbf{J} : \mathcal{P}D_G \mapsto L(G)^{\star}$ (where $L(G)^{\star}$ denotes the dual of the Lie algebra $L(G)$ of G and where $\mathcal{P}D_G$ denotes the essential G-smooth part of $\mathcal{P}(\mathcal{H})$) is equivariant [2].

Let us consider a quantum dynamical system QDS = $(\mathcal{H}, \hat{H})$ such that the associated classical dynamical system CDS[QDS] is integrable [3].

Remark I.4

It has been shown in [4] that the Aharonov-Anandan phase (i.e. the non-adiabatic quantum Berry's phase) of QDS may be related to the non-adiabatic Hannay angle [5] (i.e. the holonomy of the Hannay-Berry connection [6], [4]) of CDS[QDS].

II. A CLASSICAL HAMILTONIAN DYNAMICAL SYSTEM SEEN MATHEMATICALLY AS A QUANTUM DYNAMICAL SYSTEM

Let us start from the following [7]:

DEFINITION II.1

continuous-time classical dynamical system:
a couple $((X\,,\,\sigma\,,\,\mu), \{T_t\}_{t \in \mathbb{R}})$ such that:

- $(X\,,\,\sigma\,,\,\mu)$ is a classical probability space
- $\{T_t\}_{t \in \mathbb{R}}$ is a one-parameter family of automorphisms of $(X\,,\,\sigma\,,\,\mu)$, i.e.:

$$\mu \circ T_t^{-1} \;=\; \mu \;\; \forall t \in \mathbb{R}$$

We have seen in the previous section particular instances of the following notion:

DEFINITION II.2

classical hamiltonian dynamical system
a couple $((M, \omega), H)$ such that:

- (M, ω) is a symplectic manifold
- $H \in C^{\infty}(M)$

One has that:

Theorem II.1

HP:

$$((M, \omega), H) \text{ classical hamiltonian dynamical system such that M is compact and orientable}$$

TH:

$$((M, \omega), H) \text{ is a continuous-time classical dynamical system}$$

PROOF:

Let us introduce the classical probability space $(M, \sigma_{Borel}, \mu_{Liouville})$, where σ_{Borel} is the Borel-σ-algebra of M and where:

$$\mu_{Liouville} \;:=\; \frac{\wedge_{i=1}^{\frac{dim M}{2}} \omega}{\int_M \wedge_{i=1}^{\frac{dim M}{2}} \omega} \tag{2.1}$$

is the normalized Liouville measure over (M, σ_{Borel}).

The hamiltonian flow $\{T_t^{(H)}\}_{t \in \mathbb{R}}$ generated by H is a one-parameter family of symplectomorphisms of (M, ω) and hence:

$$\mu_{Liouville} \circ (T_t^{(H)})^{-1} \;=\; \mu_{Liouville} \;\; \forall t \in \mathbb{R} \tag{2.2}$$

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Given a continuous-time classical dynamical system $CDS := ((X\,,\,\sigma\,,\,\mu), \{T_t\}_{t \in \mathbb{R}})$ we can adopt Koopman's formalism to introduce the following:

DEFINITION II.3

quantum dynamical system associated to CDS

$$QDS[CDS] \ := \ (\mathcal{H}, \hat{H}) \tag{2.3}$$

where:

- $$\mathcal{H} \ := \ L^2(X, \mu)$$
- $\hat{H}$ is the generator (defined by Stone's Theorem [8]) of the strongly-continuous unitary group $\{\hat{U}_t = \exp(it\hat{H})\}_{t \in \mathbb{R}}$ such that:

$$(\hat{U}_t \psi)(x) \ := \ (\psi \circ T_t)(x) \ \ \psi \in \mathcal{H}, t \in \mathbb{R} \tag{2.4}$$

Remark II.1

QDS [CDS] has not to be confused with the quantum dynamical system obtained quantizing CDS since obviously:

$$\lim_{\hbar \rightarrow 0} QDS[CDS] \ \neq \ CDS \ \ \forall CDS \tag{2.5}$$

III. CONSIDERING THE OPPOSITE OF THE REMARK I.4

In the remark I.4 we saw that the Aharonov-Anandan phase of a quantum dynamical system QDS may be related to the non-adiabatic Hannay angle of CDS[QDS].

In this section we will show that also the opposite occurs, i.e. that the non-adiabatic Hannay's angle of an integrable classical hamiltonian dynamical system CDS may be related to the Aharonov-Anandan phase of QDS[CDS] [1]

Given an integrable hamiltonian classical dynamical system $((M,\omega),H)$ with (dim M = 2n) Liouville's theorem [3] states that a compact and connected level set of n independent first integrals in involution is diffeomorphic to an n-dimensional torus T^n on which the dynamics can be expressed in the action-angle canonical (i.e. such that $\omega \; = \; d\mathbf{I} \wedge d\mathbf{\Phi}$) variables $(\mathbf{I} = (I_1, \cdots, I_n)\,,\, \mathbf{\Phi} = (\Phi_1, \cdots, \Phi_n))$ as:

$$\dot{\mathbf{I}} \; = \; \mathbf{0} \tag{3.1}$$

$$\dot{\mathbf{\Phi}} \; = \; \mathbf{\Omega}(\mathbf{I}) \tag{3.2}$$

where:

$$\mathbf{\Omega}(\mathbf{I}) \; := \; \frac{\partial H(\mathbf{I})}{\partial \mathbf{I}} \tag{3.3}$$

As it is well known there are two cases:

- if $(\mathbf{k} \cdot \mathbf{\Omega} := \sum_{i=1}^{n} k_i \Omega_i \; = \; 0 \; \Rightarrow \; \mathbf{k} = \mathbf{0}) \; \forall \mathbf{k} \in \mathbb{Z}^n$ then the torus T^n is said *nonresonant* and the dynamics on it is quasi-periodic
- if $(\mathbf{k} \cdot \mathbf{\Omega} \; = \; 0 \; \nRightarrow \; \mathbf{k} = \mathbf{0}) \; \forall \mathbf{k} \in \mathbb{Z}^n$ then the torus T^n is said *resonant* and the dynamics on it is periodic

We will assume that CDS is everywhere non-degenerated, i.e.:

$$det \frac{\partial \mathbf{\Omega}}{\partial \mathbf{I}} \; \neq \; 0 \tag{3.4}$$

Remark III.1

In general the canonical coordinates $(\mathbf{I}, \mathbf{\Phi})$ are defined only locally.

This means that considered two different level sets of the n independent first-integrals in involution one obtains two different local charts $A := (U_A, \chi_A)$ and $B := (U_B, \chi_B)$ such that :

$$\chi_A(y) \; = \; (\mathbf{I}_A, \mathbf{\Phi}_A) \; : \; \omega(y) \; = \; d\mathbf{I}_A \wedge d\mathbf{\Phi}_A \;\; \forall y \in U_A \tag{3.5}$$

$$\chi_B(y) \; = \; (\mathbf{I}_B, \mathbf{\Phi}_B) \; : \; \omega(y) \; = \; d\mathbf{I}_B \wedge d\mathbf{\Phi}_B \;\; \forall y \in U_B \tag{3.6}$$

and where the map $\psi_{A,B} : \chi_B(U_A \cap U_B) \mapsto \chi_A(U_A \cap U_B)$:

$$\psi_{A,B} \; := \; \chi_A \circ \chi_B^{-1} \tag{3.7}$$

is infinitely differentiable.

Since the consideration of a symplectic atlas of charts on (M, ω) defining a collection of different action-angle variables simply complicates the situation without adding any further insight (at least for the matter we are going to discuss) we will assume that the canonical action-angle coordinates $(\mathbf{I}, \mathbf{\Phi})$ can be extended globally over (M, ω).

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[1] The Aharonov-Anandan phase of QDS[CDS] was first proposed in [9] by the author as the definition of a non adiabatic analogous of Hannay's angle. At that time I was unaware that non-adiabatic Hannay's angle was a notion already existing [5]. I strongly apologize for such an error. In this paper non-adiabatic Hannay's angle refers to the notion discovered in [5] mathematically expressed by the holonomy of the Hannay-Berry connection [6], [4], [10].

Clearly one has that:

$$QDS[CDS] \;=\; (\mathcal{H}, \hat{H}) \tag{3.8}$$

where:

$$\mathcal{H} \;=\; L^2(T^n, \frac{d\mathbf{\Phi}}{(2\pi)^n}) \tag{3.9}$$

while the strongly continuous unitary group $\{\exp(i\hat{H}t)\}_{t\in\mathbb{R}}$ is specified by its action on the following basis:

$$\mathbb{E} \;:=\; \{|\mathbf{n}>:= \exp(i\mathbf{n}\cdot\mathbf{\Phi}) \;,\; \mathbf{n}\in\mathbb{Z}^n\} \tag{3.10}$$

given by:

$$\exp(i\hat{H}t)|\mathbf{n}> = \;\exp(i\mathbf{n}\cdot\mathbf{\Omega}t)|\mathbf{n}> \quad \forall t\in\mathbb{R} \tag{3.11}$$

Considered the $U(1)$-principal bundle $\mathcal{S}(\mathcal{H})(\mathcal{P}(\mathcal{H}), U(1))$ it is well-known that the Aharonov-Anandan geometric phase is given by the holonomy of the following natural connection one-form $\mathcal{A}\in T^\star\mathcal{P}(\mathcal{H})\otimes L[U(1)]$ (where we denote by L[G] the Lie algebra of a Lie group G):

$$\mathcal{A}_\psi(X) \;:=\; iIm<\psi|X> \quad \psi\in\mathcal{S}(\mathcal{H}), X\in T_\psi\mathcal{S}(\mathcal{H})\subset\mathcal{H} \tag{3.12}$$

A curve $t\mapsto|\psi(t)>\in\mathcal{S}(\mathcal{H})$ is horizontal with respect to $\mathcal{A}$ if and only if:

$$<\psi(t)|\dot{\psi}(t)> = \;0 \;\;\forall t \tag{3.13}$$

So the Aharonov-Anandan geometric phase acquired by QDS[CDS] when it is subjected to a loop $\gamma:[0,1]\mapsto\mathcal{P}(\mathcal{H})$ such that $\gamma(0)=\gamma(1)=P_\psi$, $|\psi>\in\mathcal{S}(\mathcal{H})$ is the holonomy $\tau_\gamma^{\mathcal{A}}(|\psi>)$.

Let us now consider a family of integrable classical hamiltonian dynamical systems $CDS_x \;:=\; ((M,\omega), H_x)$ where x is a parameter taking values on a parameters' connected differentiable manifold P such that H_x depends smoothly by x and it there exists a point $x_0\in P$ such that $CDS_{x_0} \;=\; CDS$.

Let us then introduce the family of quantum dynamical systems:

$$QDS[CDS_x] \;=:\; (\mathcal{H}, \hat{H}_x) \;\; x\in P \tag{3.14}$$

Let us suppose that the parameter x evolves adiabatically realizing a loop $\gamma:[0,1]\mapsto P:\gamma(0)=\gamma(1)=x_0$ in P.

The adiabatic limit under which the Aharonov-Anandan phase of QDS[CDS] reduces to the adiabatic Berry phase of such a quantum dynamical system may be simply implemented through a suitable pullback [11].

In the adiabatic limit the basis:

$$\mathbb{E}_x \;:=\; \{\,|\mathbf{n},x> \quad \mathbf{n}\in\mathbb{Z}^n\,, x\in P\} \tag{3.15}$$

continues to be formed by eigenvectors of $\hat{U}_t$.

Let us assume that the eigenvalue corresponding to $|\mathbf{n},x>$ is non-degenerate for every $x\in P$.

Given $\mathbf{n}\in\mathbb{Z}^n$ let us then introduce the following map $f_{\mathbf{n}}:P\;\mapsto\;\mathcal{P}(\mathcal{H})$:

$$f_{\mathbf{n}}(x) \;:=\; P_{|\mathbf{n},x>} \;=\; |\mathbf{n},x><\mathbf{n},x| \tag{3.16}$$

Let us then introduce the pullback-bundle $f_{\mathbf{n}}^\star\mathcal{S}(\mathcal{H})$ of the U(1)-bundle $\mathcal{S}(\mathcal{H})(\mathcal{P}(\mathcal{H}), U(1))$ by $f_{\mathbf{n}}$ and let us denote by $f_{\mathbf{n}}^\star\mathcal{A}$ the connection on the principal bundle $f_{\mathbf{n}}^\star\mathcal{S}(\mathcal{H})$ induced by the connection $\mathcal{A}$ through the pull-back operation; clearly such a connection is the Berry-Simon connection.

The adiabatic Berry phase developed by QDS[CDS] after the adiabatic evolution γ is then the holonomy $\tau_\gamma^{f_{\mathbf{n}}^\star\mathcal{A}}$ of the connection $f_{\mathbf{n}}^\star\mathcal{A}$ along the loop γ.

Let us now consider the Hannay angles of the classical hamiltonian dynamical system CDS.

At this purpose let us introduce $E:=M\times P$ and the trivial bundle $E\overset{\pi_P}{\to}P$ where clearly $\pi_P:M\times P\mapsto P$ is such that:

$$\pi_P(y,x) \;:=\; x \;\; y\in M, x\in P \tag{3.17}$$

and let us introduce also the other canonical projection $\pi_M:M\times P\mapsto P$ defined as:

$$\pi_M(y,x) \;:=\; y \;\; y\in M, x\in P \tag{3.18}$$

Let us observe that the restriction of the pullback $\pi_M^\star \omega$ to each fibre $E_x \; := \; \pi_P^{-1}(x)$ is a symplectic form on such a fibre.

Introduced the natural splitting of the total exterior derivative on $M \times P$ of a function $f \in C^\infty(M \times P)$:

$$df \; = \; d_M f + d_P f \tag{3.19}$$

meaning that, if $(y^1, \cdots, y^{2n})$ are local coordinates on M and $(x^1, \cdots, x^m)$ are local coordinates on P, then:

$$d_M f \; = \; \sum_{i=1}^{2n} \frac{\partial f}{\partial y^i} \, dy^i \tag{3.20}$$

$$d_P f \; = \; \sum_{i=1}^{m} \frac{\partial f}{\partial x^i} \, dx^i \tag{3.21}$$

let us introduce the following:

DEFINITION III.1

fibrewise hamiltonian vector field X_f corresponding to f:

$$i_{X_f}(\pi_M^\star \omega) \; = \; d_M f \tag{3.22}$$

Note that X_f is tangent to each fibre $\pi_P^{-1}(x)$ and hence defines an hamiltonian vector field on $\pi_P^{-1}(x)$ in the usual sense.

Given a Lie group G:

DEFINITION III.2

family of hamiltonian G-actions on E

a smooth left action $\Upsilon : G \times E \mapsto E$ of G on E such that:

- each fibre E_x is invariant under the action
- the action, restricted to each fibre E_x, is symplectic
- it admits a smooth family of momentum maps $\mathbf{J} : M \times P \mapsto L(G)^\star$, i.e., for any $x \in P$, the map $\mathbf{J}(\cdot, x) : M \mapsto L(G)^\star$ is a momentum map in the usual sense for every $x \in P$.

Given a family $\Upsilon : G \times E \mapsto E$ of hamiltonian G-actions on E and an arbitrary tensor T on E let us introduce the following:

DEFINITION III.3

G-average of T:

$$< T > := \; \frac{1}{|G|} \int_G \Upsilon_g^\star T \, dg \tag{3.23}$$

where dg is the Haar measure on G and where $|G| := \int_G dg$.

Let us now observe that since CDS_x is integrable for every $x \in P$ there exists, due to Liouville theorem, a set of local x-dependent action variables $\mathbf{I}(\cdot \, , \, x) := (I_1(\cdot \, , \, x), \cdots, I_n(\cdot \, , \, x))$.

For the same reasons exposed in the remark III.1 we will assume, from here and beyond, that this system is globally defined on E and, furthermore, that is everywhere non-degenerated, i.e.:

$$det \frac{\partial \mathbf{\Omega}}{\partial \mathbf{I}} \; \neq \; 0 \tag{3.24}$$

Let us now look at the n-torus T^n as an abelian Lie group; we have clearly that:

$$L(T^n)^\star \; = \; L(T^n) \; = \; \mathbb{R} \tag{3.25}$$

Under the assumed hypotheses it results defined a family of hamiltonian $T^{(n)}$-actions $\Upsilon : T^n \times E \mapsto E$ on E whose associated smooth family of momentum maps is $\mathbf{J} \; = \; \mathbf{I} : E \mapsto \mathbb{R}^n$.

Remark III.2

Let us observe that chosen at random an initial condition on M the probability of getting into a resonant torus is zero.

Since the quasi-periodic dynamics on a non-resonant torus is ergodic, the T^n-average and the temporal averages are equal.

Let us introduce the following:

DEFINITION III.4

Hannay-Berry connection on $E \stackrel{\pi_P}{\to} P$

the connection $\mathcal{B}$ on $E \stackrel{\pi_P}{\to} P$ such that:

$$hor_{\mathcal{B}}(X) \ := < (0,X) > \quad \forall X \in T_x P \tag{3.26}$$

where $hor_{\mathcal{B}}(X) \in T_x P \times T_y M$ is the horizontal lift of a vector $X \in T_x P$ induced by the connection $\mathcal{B}$.

Let $\mu \in \mathbb{R}^n$ be a regular value of the momentum map $\mathbf{J}(\cdot, x) : M \mapsto \mathbb{R}^n$ and let us introduce the following sets:

$$E_x^\mu \ := \ \mathbf{J}^{-1}(\mu) \cap \pi_P^{-1}(x) \ = \ T^n \tag{3.27}$$

$$E^\mu \ := \ \cup_{x \in P} E_x^\mu \tag{3.28}$$

Introducing also the projection:

$$\pi_\mu \ := \ \pi_P|_{E^\mu} \tag{3.29}$$

one has that $E^\mu(P, T^n)$ is a torus-bundle over M [2].

Let us finally introduce the following:

DEFINITION III.5

Hannay-Berry connection on $E^\mu(P, T^n)$

the restriction of $\mathcal{B}$ to E^μ.

Let us suppose that the parameter x evolves realizing a loop $\gamma : [0,1] \mapsto P : \gamma(0) = \gamma(1) = x_0$ in P.

The Hannay angle of CDS is then the holonomy $\tau_\gamma^{\mathcal{B}}$.

Let us now compare the Aharonov-Anandan phase of QDS[CDS] and the Hannay angle of CDS.

As we saw the former is the holonomy $\tau_\gamma^{\mathcal{A}}$ over the U(1)-bundle $\mathcal{S}(\mathcal{H})(\mathcal{P}(\mathcal{H}), U(1))$ while the latter is the holonomy $\tau_\gamma^{\mathcal{B}}$ over the T^n-bundle $E^\mu(P, T^n)$.

Let us first of all make the passage to the Simon's spectral bundle considering, for each $\mathbf{n} \in \mathbb{Z}^n$, the map $f_{\mathbf{n}} : P \mapsto \mathcal{P}(\mathcal{H})$:

$$f_{\mathbf{n}}(x) \ := \ P_{|\mathbf{n},x>} \ = \ |\mathbf{n}, x >< \mathbf{n}, x| \tag{3.30}$$

and taking into account the spectral bundle $F := f_{\mathbf{n}}^\star \mathcal{S}(\mathcal{H})$ previously introduced:

such a U(1)-bundle has the same base space, i.e. P , of the T^n-bundle $E^\mu(P, T^n)$ while its fibre F_x in $x \in P$ is:

$$F_x \ = \ \{\exp(i\alpha)|\mathbf{n}, x >, \alpha \in \mathbb{R}\} \tag{3.31}$$

Given $\mathbf{n} \in \mathbb{Z}^n$ let us now introduce the following:

DEFINITION III.6

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[2] The first intuitive idea of the fact that the adiabatic Hannay angle should have been given by the holonomy of a connection on such a torus-bundle was first proposed in [12]

map of relation between the Hannay angle of CDS and the Aharonov-Anandan phase of QDS[CDS]:
the map $R_{\mathbf{n}} : Hol_{\mathcal{B}} \mapsto Hol_{f_{\mathbf{n}}^{\star}\mathcal{A}}$:

$$R_{\mathbf{n}}(\tau_{\gamma}^{\mathcal{B}}) \; = \; \tau_{\gamma}^{f_{\mathbf{n}}^{\star}\mathcal{A}} \;\; \forall \gamma \in C_{x_0}(P) \tag{3.32}$$

where $Hol_{\mathcal{B}}$ is the holonomy group of the connection $\mathcal{B}$, where $Hol_{f_{\mathbf{n}}^{\star}\mathcal{A}}$ is the holonomy group of the connection $f_{\mathbf{n}}^{\star}\mathcal{A}$ and where:

$$C_{x_0}(P) \; := \; \{\gamma : [0,1] \mapsto P \; : \; \gamma(0) = \gamma(1) \; = \; x_0\} \tag{3.33}$$

is the set of loops in P based at x_0.

Remark III.3

Let us observe that $R_{\mathbf{n}}$ maps the Hannay angle of CDS into the adiabatic Berry phase of QDS[CDS].

Since the adiabatic Berry phase of QDS[CDS] is a particular case of the Aharonov-Anandan phase related to it by the pull-back $f_{\mathbf{n}}^{\star}$ we can see R as a map relating the Hannay angle of CDS and the Aharonov-Anandan phase of QDS[CDS].

The function $R_{\mathbf{n}}$ maps the holonomy of $\mathcal{B}$ associated to a loop γ into the holonomy of $f_{\mathbf{n}}^{\star}\mathcal{A}$ associated to the same loop.

Since $\tau_{\gamma}^{\mathcal{B}} \in T^n$ while $\tau_{\gamma}^{f_{\mathbf{n}}^{\star}\mathcal{A}} \in U(1)$ the map $R_{\mathbf{n}}$ has to be of the form:

$$R_{\mathbf{n}}(\tau_{\gamma}^{\mathcal{B}}) \; = \; \exp[iS(\mathbf{n} \cdot \tau_{\gamma}^{\mathcal{B}})] \tag{3.34}$$

for some $S : \mathbb{R} \mapsto \mathbb{R}$.

Considering the case in which $\bar{\gamma}$ is the constant loop $\bar{\gamma}(t) \; := \; x_0 \;\; \forall t \in [0,1]$ one has that since $\tau_{\bar{\gamma}}^{\mathcal{B}} = \mathbb{I}_{T^n}$ and $\tau_{\bar{\gamma}}^{f_{\mathbf{n}}^{\star}\mathcal{A}} = \mathbb{I}_{U(1)}$ it follows that:

$$S(0) \; = \; 0 \tag{3.35}$$

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